Exponential Functions (GROWTH)

In everyday conversation the phrase "growing exponentially" is used to describe any situation where some quantity is increasing rapidly with the passage of time. But in mathematics, the terms *exponential growth* and *exponential decay* refer to particular important patterns of change.
For example, when wildlife biologists estimated the population of gray wolves in Michigan, Wisconsin, and Minnesota, they found it growing exponentially—at an annual rate of about 25% from a base of about 170 wolves in 1990 to about 3,100 wolves in 2003.
In this unit, you will develop understanding and skill required to study patterns of change like growth of the mid-western gray wolf population and decay of medicines in the human body.
The key ideas and strategies for studying those patterns will be developed in two lessons

*I*n the popular book and movie *Pay It Forward*, 12-year-old Trevor McKinney gets a challenging assignment from his social studies teacher.

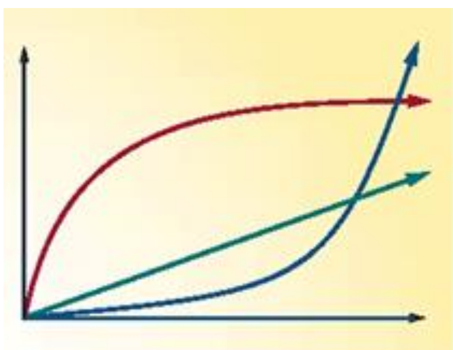
Think of an idea for world change, and put it into practice!

Trevor came up with an idea that fascinated his mother, his teacher, and his classmates. He suggested that he would do something really good for three people. Then when they ask how they can pay him back for the good deeds, he would tell them to "pay it forward"—each doing something good for three other people.
Trevor figured that those three people would do something good for a total of nine others. Those nine would do something good for 27 others, and so on. He was sure that before long there would be good things happening to billions of people all around the world.

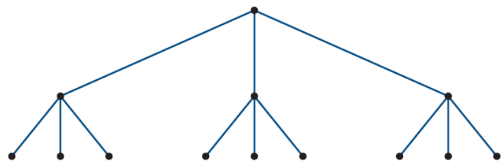
a. How many people would receive a Pay It Forward good deed at each of the next several stages of the process?

b. What is your best guess about the number of people who would receive Pay It Forward good deeds at the tenth stage of the process?

c. Which of the graphs at the right do you think is most likely to represent the pattern by which the number of people receiving Pay It Forward good deeds increases as the process continues over time?



The number of good deeds in the Pay It Forward pattern can be represented by a *tree graph* that starts like this



The vertices represent the people who receive and do good deeds. Each edge represents a good deed done by one person for another. As you work on the problems of this investigation, look for answers to these questions:
What are the basic patterns of exponential growth in variations of the Pay It Forward process?
How can those patterns be expressed with symbolic rules?

1. At the start of the Pay It Forward process, only one person does good deeds—for three new people. In the next stage, the three new people each do good things for three more new people. In the next stage, nine people each do good things for three more new people, and so on, with no person receiving more than one good deed.

a. Make a table that shows the number of people who will receive good deeds at each of the next seven stages of the Pay It Forward process. Then plot the (*stage, number of good deeds*) data.

Stage of Process	1	2	3	4	5	6	7	8	9	10
Number of Good Deeds	3	9	27							

b. How does the number of good deeds at each stage grow as the tree progresses? How is that pattern of change shown in the plot of the data?

c. How many stages of the Pay It Forward process will be needed before a total of at least 25,000 good deeds will be done?

2. Consider now how the number of good deeds would grow if each person touched by the Pay It Forward process were to do good deeds for only two other new people, instead of three.

a. Make a tree graph for several stages of this Pay It Forward process.

b. Make a table showing the number of good deeds done at each of the first 10 stages of the process and plot those sample (*stage, number of good deeds*) values.

Stage of Process	1	2	3	4	5	6	7	8	9	10
Number of Good Deeds										

c. How does the number of good deeds increase as the Pay It Forward process progresses in stages? How is that pattern of change shown in the plot of the data?

d. How many stages of this process will be needed before a total of 25,000 good deeds will have been done?

3. In the two versions of Pay It Forward that you have studied, you can use the number of good deeds at one stage to calculate the number at the next stage.

a. Write a recursive rule that express the two patterns.

b. How do the numbers and calculations indicated in the rules express the patterns of change in tables of (*stage, number of good deeds*) data?

c. Write a recursive rule that that could be used to model a Pay It Forward process in which each person does good deeds for four other new people.

d. What pattern of change would you expect to see in a table of (*stage, number of good deeds*) data for this Pay It Forward process?

4. What are the main steps (not keystrokes) required to use a calculator to produce tables of values like those you made in Problems 1 and 2?

5. It is also convenient to have rules that will give the number of good deeds N at any stage x of the Pay It Forward process, without finding all the numbers along the way to stage x. When students in one class were given the task of finding such a rule for the process in which each person does three good deeds for others, they came up with four different ideas:

$N = 3x$

$N = x + 3$

$N = 3^x$

$N = 3x + 1$

a. Are any of these rules for predicting the number of good deeds N correct? How do you know?

b. Write an "$N = \ldots$" rule that would show the number of good deeds at stage number x if each person in the process does good deeds for two others.

c. Write an "$N = \ldots$" rule that would show the number of good deeds at stage number x if each person in the process does good deeds for four others.

1. Infections seldom start with a single bacterium. Suppose that you cut yourself on a rusty nail that puts 25 bacteria cells into the wound. Suppose also that those bacteria divide in two after every quarter of an hour.

a. Make and record a guess of how many bacteria you think would be present in the cut after 8 hours (32 quarter-hours) if the infection continues to spread as predicted. (Assume that your body does not fight off the infection and you do not apply medication.) Then answer the following questions to check your ability to estimate the rate of exponential growth.

Number of Quarter-Hour Periods	0	1	2	3	4
Number of Bacteria in the Cut	25	50			

b. Write a recursive rule showing how the number of bacteria changes from one quarter-hour to the next, starting from 25 at time 0.

c. Write a rule showing how to calculate the number of bacteria N in the cut after x quarter-hour time periods.

d Use the rule in either Parts b or c to calculate the number of bacteria after 8 hours. Then compare the results to each other and to your initial estimate in Part a.

2. Investigate the number of bacteria expected after 8 hours if the starting number of bacteria is **30.**

a. Make a table of (*number of time periods*, *bacteria count*) values for 8 quarter-hour time periods.

Number of Quarter-Hour Periods	0	1	2	3	4
Number of Bacteria in the Cut					

b. Write a recursive rule showing how the number of bacteria changes from one quarter-hour to the next, starting from 25 at time 0.

c. Write a rule showing how to calculate the number of bacteria *N* in the cut after *x* quarter-hour time periods.

d Use the rule in either Parts b or c to calculate the number of bacteria after 8 hours. Then compare the results to each other and to your initial estimate in Part a.

3. Investigate the number of bacteria expected after 8 hours if the starting number of bacteria is **40**.

a. Make a table of (*number of time periods*, *bacteria count*) values for 8 quarter-hour time periods.

Number of Quarter-Hour Periods	0	1	2	3	4
Number of Bacteria in the Cut					

b. Write a recursive rule showing how the number of bacteria changes from one quarter-hour to the next, starting from 25 at time 0.

c. Write a rule showing how to calculate the number of bacteria *N* in the cut after *x* quarter-hour time periods.

d Use the rule in either Parts b or c to calculate the number of bacteria after 8 hours. Then compare the results to each other and to your initial estimate in Part a.

4. Investigate the number of bacteria expected after 8 hours if the starting number of bacteria is **60.**

a. Make a table of (*number of time periods, bacteria count*) values for 8 quarter-hour time periods.

Number of Quarter-Hour Periods	0	1	2	3	4
Number of Bacteria in the Cut					

b. Write a recursive rule showing how the number of bacteria changes from one quarter-hour to the next, starting from 25 at time 0.

c. Write a rule showing how to calculate the number of bacteria *N* in the cut after *x* quarter-hour time periods.

d Use the rule in either Parts b or c to calculate the number of bacteria after 8 hours. Then compare the results to each other and to your initial estimate in Part a.

5. Investigate the number of bacteria expected after 8 hours if the starting number of bacteria is **100**.

a. Make a table of (*number of time periods, bacteria count*) values for 8 quarter-hour time periods.

Number of Quarter-Hour Periods	0	1	2	3	4
Number of Bacteria in the Cut					

b. Write a recursive rule showing how the number of bacteria changes from one quarter-hour to the next, starting from 25 at time 0.

c. Write a rule showing how to calculate the number of bacteria *N* in the cut after *x* quarter-hour time periods.

d Use the rule in either Parts b or c to calculate the number of bacteria after 8 hours. Then compare the results to each other and to your initial estimate in Part a.

6. Just as bacteria growth won't always start with a single cell, other exponential growth processes can start with different initial numbers. Think again about the Pay It Forward scheme in Investigation 1.

Suppose that four good friends decide to start their own Pay It Forward tree. To start the tree, they each do good deeds for three different people. Each of those new people in the tree does good deeds for three other new people, and so on.

a. What *recursive* rule shows how to calculate the number of good deeds done at each stage of this tree?

b. What "N = ..." rule shows how to calculate the number of good deeds done at any stage x of this tree?

c. How would the *recursive rule* and "N = ..." rules be different if the group of friends starting the tree had five members instead of four?

d. Which of the Pay It Forward schemes below would most quickly reach a stage in which 1,000 good deeds are done? Why does that make sense?

Scheme 1: Start with a group of four friends and have each person in the tree do good deeds for two different people; or

Scheme 2: Start with only two friends and have each person in the tree do good deeds for three other new people.

In studying exponential growth, it is helpful to know the *initial value* of the growing quantity. For example, the initial value of the growing bacteria population in Problem 1 was 25. You also need to know when the initial value occurs. For example, the bacteria population was 25 after 0 quarter- hour periods.

7. In Problem 6 on the other hand, 12 good deeds are done at Stage 1. In this context, "Stage 0" does not make much sense, but we can extend the pattern backward to reason that $N = 4$ when $x = 0$.

   Use your calculator key to find each of the following values: 2^0, 3^0, 5^0, 23^0.

a. What seems to be the calculator value for b^0, for any positive value of b?

b. Recall the examples of exponential patterns in bacterial growth. How do the "$N = ...$" rules for those situations make the calculator output for b^0 reasonable?

8. Now use your calculator to make tables of (x, y) values for each of the following functions. Use integer values for x from 0 to 6. Make notes of your observations and discussion of questions in Parts a and b.

i. $y = 5(2^x)$ **ii.** $y = 4(3^x)$ **iii.** $y = 3(5^x)$ **iv.** $y = 7(2.5^x)$

a. What patterns do you see in the tables? How do the patterns depend on the numbers in the function rule?

b. What differences would you expect to see in tables of values and graphs of the two exponential functions $y = 3(6^x)$ and $y = 6(3^x)$?

9. Suppose you are on a team studying the growth of bacteria in a laboratory experiment. At the start of your work shift in the lab, there are 64 bacteria in one petri dish culture, and the population seems to be doubling every hour.

a. What rule should predict the number of bacteria in the culture at a time x hours after the start of your work shift?

b. What would it mean to calculate values of y for negative values of x in this situation?

c. What value of y would you expect for $x = -1$? For $x = -2$? For $x = -3$ and -4?

d. Use your calculator to examine a table of (x, y) values for the function $y = 64(2^x)$ when $x = 0, -1, -2, -3, -4, -5, -6$.

Find solutions for each of the following equations and inequalities. In each case, be prepared to explain what the solution tells about bacteria growth in the

a. $1{,}024 = 64(2^x)$

b. $8{,}192 = 64(2^x)$

c. $64(2^x) > 25{,}000$

d. $4 = 64(2^x)$

e. $64(2^x) < 5{,}000$

f. $64(2^x) = 32$

The drug penicillin was discovered by observation of mold growing on biology lab dishes. Suppose a mold begins growing on a lab dish. When first observed, the mold covers 7 cm^2 of the dish surface, but it appears to double in area every day.

a. What rules can be used to predict the area of the mold patch 4 days after the first measurement:

i. recursive rule

ii. "y = ..." form

b. How would each rule in Part a change if the initial mold area was only 3 cm^2?

c. How would each rule in Part a change if the area of the mold patch increased by a factor of 1.5 every day?

d. What mold area would be predicted after 5 days in each set of conditions from Parts a-c?

e. For "y = ..." rules used in calculating growth of mold area, what would it mean to calculate values of y when x is a negative number?

f. Write and solve equations or inequalities that help to answer these questions.

i. If the area of a mold patch is first measured to be 5 cm^2 and the area doubles each day, how long will it take that mold sample to grow to an area of 40 cm^2?

ii. For how many days will the mold patch in part i have an area less than 330 cm^2?

Every now and then you may hear about somebody winning a big payoff in a state lottery. The winnings can be 1, 2, 5, or even 100 million dollars. The big money wins are usually paid off in annual installments for about 20 years. But some smaller prizes are paid at once. How would you react if this news report were actually about you?

Kalamazoo Teen Wins Big Lottery Prize

A Kalamazoo teenager has just won the daily lottery from a Michigan lottery ticket that she got as a birthday gift from her uncle. In a new lottery payoff scheme, the teen has two payoff choices.

One option is to receive a single $10,000 payment now.

In the other plan, the lottery promises a single payment of $20,000 ten years from now.

1. Imagine that you had just won that Michigan lottery prize.

a. Discuss with others your thinking on which of the two payoff methods to choose.

b. Suppose a local bank called and said you could invest your $10,000 payment in a special 10-year certificate of deposit (CD), earning 8% interest compounded yearly. How would this affect your choice of payoff method?

As you work on the problems of this investigation, look for answers to the question

How can you represent and reason about functions involved in investments paying compound interest?

Of the two lottery payoff methods, one has a value of $20,000 at the end of 10 years. The value (in 10 years) of receiving the $10,000 payoff now and putting it in a 10-year certificate of deposit paying 8% interest compounded annually is not so obvious.

- After one year, your balance will be: 10,000 + (0.08 × 10,000) = 1.08 × 10,000 = $10,800.
- After the second year, your balance will be: 10,800 + (0.08 × 10,800) = 1.08 × 10,800 = $11,664.

During the next year, the CD balance will increase in the same way, starting from $11,664, and so on.

2. Write rules that will allow you to calculate the balance of this certificate of deposit:

a. for the next year, using the balance from the current year.

b. after any number of years x.

3. Use the rules from Problem 2 to determine the value of the certificate of deposit after 10 years. Then decide which 10-year plan will result in more money and how much more money that plan will provide.

4. Look for an explanation of your conclusion in Problem 3 by answering these questions about the potential value of the CD paying 8% interest compounded yearly.

a. Describe the pattern of growth in the CD balance as time passes.

b. Why isn't the change in the CD balance the same each year?

c. How is the pattern of increase in CD balance shown in the shape of a graph for the function relating CD balance to time?

d. How could the pattern of increase have been predicted by thinking about the rules (*recursive* and "$y = ...$") relating CD balance to time?

5. Suppose that the prize winner decided to leave the money in the CD, earning 8% interest for more than 10 years. Use tables or graphs to estimate solutions for the following equations and inequalities. In each case, be prepared to explain what the solution tells about the growth of a $10,000 investment that earns 8% interest compounded annually.

a. $10{,}000(1.08^x) = 25{,}000$

b. $10{,}000(1.08^x) = 37{,}000$

c. $10{,}000(1.08^x) = 50{,}000$

d. $10{,}000(1.08^x) \geq 25{,}000$

e. $10{,}000(1.08^x) \leq 30{,}000$

f. $10{,}000(1.08^x) = 10{,}000$

6. Compare the pattern of change and the final account balance for the plan that invests $10,000 in a CD that earns 8% interest compounded annually over 10 years to those for the following possible savings plans over 10 years. Write a summary of your findings.

a. Initial investment of $15,000 earning only 4% annual interest compounded yearly

b. Initial investment of $5,000 earning 12% annual interest compounded yearly

1. Suppose that census counts of Midwest wolves began in 1990 and produced these estimates for several different years:

Time Since 1990 (in years)	0	2	5	7	10	13
Estimated Wolf Population	100	300	500	900	1,500	3,100

a. Plot the wolf population data and decide whether a linear or exponential function seems likely to match the pattern of growth well. For the function type of your choice, experiment with different rules to see which rule provides a good model of the growth pattern.

b. Use your calculator or computer software to find both linear and exponential regression models for the given data pattern. Compare the fit of each function to the function you developed by experimentation in Part a.

c. What do the numbers in the linear and exponential function rules from Part b suggest about the pattern of change in the wolf population?

d. Use the model for wolf population growth that you believe to be best to calculate population estimates for the missing years 1994 and 2001 and then for the years 2015 and 2020.

2. Suppose that census counts of Alaskan bowhead whales began in 1970 and produced these estimates for several different years:

Time Since 1970 (in years)	0	5	15	20	26	31
Estimated Whale Population	5,040	5,800	7,900	9,000	11,000	12,600

a. Plot the given whale population data and decide which type of function seems likely to match the pattern of growth well. For the function type of your choice, experiment with different rules to see which provides a good model of the growth pattern.

b. Use your calculator or computer software to find both linear and exponential regression models for the data pattern. Compare the fit of each function to that of the function you developed by experimentation in Part a.

c. What do the numbers in the linear and exponential function rules from Part b suggest about patterns of change in the whale population?

d. Use the model for whale population growth that you believe to be best to calculate population estimates for the years 2002, 2005, and 2010.

Products of Powers Work with exponents is often helped by writing products like $b^x\ b^y$ in simpler form or by breaking a calculation like b^z into a product of two smaller numbers.

1. Complete the statement below:

a. $(2^{10})(2^3)=$

b. $(5^2)(5^4)=$

c. $(3)(3^7)=$

d. $(2^x)(2^4)= 2^7$

e. $(b^4)(b^2)=$

f. $(9^x)(9^y)= 9^5$

2. Examine the results of your work on Problem 1.

a. What pattern seems to relate task and result in every case?

b. How would you use the definition of exponent or other reasoning to convince another student that your answer to Part a is correct?

3. When people work with algebraic expressions that involve exponents, there are some common errors that slip into the calculations. How would you help other students correct their understanding of operations with exponents if their work showed the following errors?

a. $3^5 = 15$

b. $3^4\ 5^2 = 15^8$

c. $3^4\ 5^2 = 15^6$

d. $3^4 + 3^2 = 3^6$

Power of a Power You know that $8 = 2^3$ and $64 = 8^2$, so $64 = (2^3)^2$. As you work on the next problems, look for a pattern suggesting how to write equivalent forms for expressions like $(b^x)^y$ that involve powers of powers.

4. Find values for x and z that will make these equations true statements:

a. $(2^3)^4 =$

b. $(3^5)^2 =$

c. $(5^2)^x =$

d. $(b^2)^5 =$

5. Examine the results of your work on Problem 4.

a. What pattern seems to relate task and result in every case?

b. How would you use the definition of exponent or other reasoning to convince another student that your answer to Part a is correct?

c. What would you expect to see as common errors in evaluating a power of a power like $(4^3)^2$? How would you help someone who made those errors correct their understanding of how exponents work?

Power of a Product The area of a circle can be calculated from its radius r using the formula $A = \pi r^2$. It can be calculated from the diameter using the formula $A = \pi(0.5d)^2$. Next search for a pattern showing how powers of products like $(0.5d)^2$ can be expressed in equivalent forms.

6. Find values for x and y that will make these equations true statements:

a. $(6 \cdot 11)^3 =$

b. $(3\pi)^4 =$

c. $(2m)^3 =$

d. $(m^3p)^2 =$

7. Examine the results of your work on Problem 6.

a. What pattern seems to relate task and result in every case?

b. How would you use the definition of exponent or other reasoning to convince another student that your answer to Part a is correct?

c. What would you expect to see as the most common errors in evaluating a power of a product like $(4t)^3$? How would you help someone who made those errors correct their understanding of how exponents work?

Exponential Functions (DECAY)

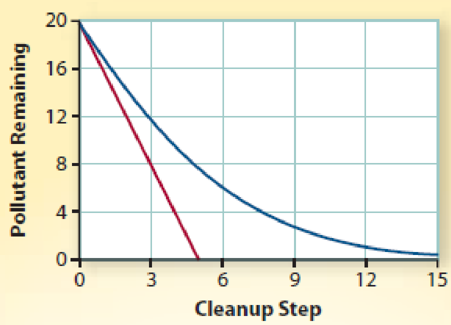
In 1989, the oil tanker Exxon Valdez ran aground in waters near the Kenai peninsula of Alaska. Over 10 million gallons of oil spread on the waters and shoreline of the area, endangering wildlife. That oil spill was eventually cleaned up—some of the oil evaporated, some was picked up by specially equipped boats, and some sank to the ocean floor as sludge. But the experience had lasting impact on thinking about environmental protection.

For scientists planning environmental cleanups, it is important to be able to predict the pattern of dispersion in such contaminating spills. Suppose that an accident dropped some pollutant into a large aquarium. It's not practical to remove all water from the aquarium at once, so the cleanup has to take place in smaller steps. A batch of polluted water is removed and replaced by clean water. Then the process is repeated.

Think about the following experiment that simulates pollution and cleanup of the aquarium.

- Mix 20 black checkers (the pollution) with 80 red checkers (the clean water).
- Remove 20 checkers from the mixture (without looking at the colors) and replace them with 20 red checkers (clean water). Record the number of black checkers remaining. Then shake the new mixture. This simulates draining off some of the polluted water and replacing it with clean water.
- In the second step, remove 20 checkers from the new mixture (without looking at the colors) and replace them with 20 red checkers (more clean water). Record the number of black checkers remaining. Then stir the new mixture.
- Repeat the remove-replace-record-mix process for several more steps.

The graphs below show two possible outcomes of the pollution and cleanup simulation.



a What pattern of change is shown by each graph?

b Which graph shows the pattern of change that you would expect for this situation? Test your idea by running the experiment and plotting the (*cleanup step*, *pollutant remaining*) data.

c What sort of function relating pollution *P* and cleanup steps *x* would you expect to match your data plot? Test your idea using a graphing calculator or computer software.

The pollution cleanup experiment gives data in a pattern that occurs in many familiar and important problem situations. That pattern is called *exponential decay*. Your work on problems of this lesson will reveal important properties and uses of exponential decay functions and fractional exponents.

Most popular American sports involve balls of some sort. In designing those balls, one of the most important factors is the bounciness or *elasticity* of the ball. For example, if a new golf ball is dropped onto a hard surface, it should rebound to about $\frac{2}{3}$ of its drop height. The pattern of change in successive rebound heights will be similar to that of the data in the pollution cleanup experiment.

As you work on the problems of this investigation, look for answers to this question:

What mathematical patterns in tables, graphs, and symbolic rules are typical of exponential decay relations?

1. Suppose a new golf ball drops downward from a height of 27 feet onto a paved parking lot and keeps bouncing up and down, again and again. Rebound height of the ball should be $\frac{2}{3}$ of its drop height. Make a table and plot of the data showing expected heights of the first ten bounces of the golf ball.

Bounce Number	0	1	2	3	4	5	6	7	8	9	10
Rebound Height (in feet)	27										

ai. How does the rebound height change from one bounce to the next?

aii. How is that pattern shown by the shape of the data plot?

b. What recursive rule shows how to calculate the rebound height for any bounce from the height of the preceding bounce?

c. What rule beginning "$y = \ldots$" shows how to calculate the rebound height after any number of bounces?

d. How will the data table, plot, and rules for calculating rebound height change if the ball drops first from only 15 feet?

As is the case with all mathematical models, data from actual tests of golf ball bouncing will not match exactly the predictions from rules about ideal bounces. You can simulate the kind of quality control testing that factories do by running some experiments in your classroom. Work with a group of

2. Get a golf ball and a tape measure or meter stick for your group. Decide on a method for measuring the height of successive rebounds after the ball is dropped from a height of at least 100 cm. Collect data on the rebound height for successive bounces of the ball.

Bounce #	0	1	2	3	4	5
Height, cm	100	57	39	22	9	4

a. Compare the pattern of your data to that of the model that predicts rebounds which are $\frac{2}{3}$ of the drop height. Would a rebound height factor other than $\frac{2}{3}$ give a better model for your data? Be prepared to explain your reasoning.

b. Write a recursive rule that relates the rebound height of any bounce of your tested ball to the height of the preceding bounce.

c. Write a rule beginning "y = ..." to predict the rebound height after any bounce.

3. Repeat the experiment of Problem 2 with racquet ball.

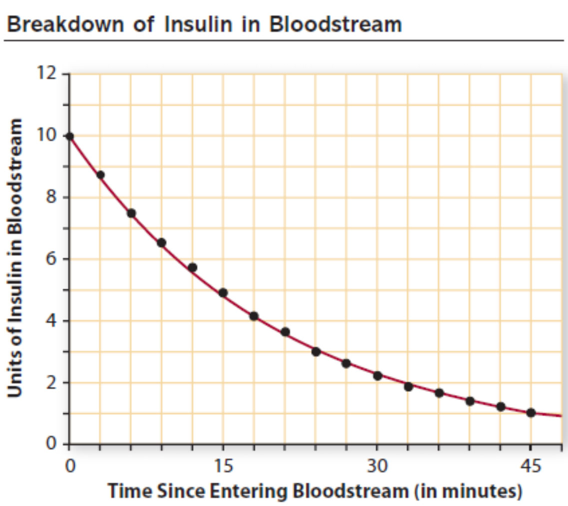
Bounce #	0	1	2	3	4	5
Height, cm	100	70	49	28	22	9

a. Study the data to find a reasonable estimate of the rebound height factor for your ball.

b. Write a recursive rule and a rule beginning "y = ..." to model the rebound height of your ball on successive bounces.

Prescription drugs are a very important part of the human health equation. Many medications are essential in preventing and curing serious physical and mental illnesses.
Diabetes, a disorder in which the body cannot metabolize glucose properly, affects people of all ages. In 2005, there were about 14.6 million diagnosed cases of diabetes in the United States. It was estimated that another 6.2 million cases remained undiagnosed.

In 5-10% of the diagnosed cases, the diabetic's body is unable to produce insulin, which is needed to process glucose. To provide this essential hormone, these diabetics must take injections of a medicine containing insulin. The medications used (called insulin delivery systems) are designed to release insulin slowly. The insulin itself breaks down rather quickly. The rate varies greatly among individuals, but the following graph shows a typical pattern of insulin decrease.



Time	Units of Insulin
0	
3	
6	
9	
12	
15	
18	
21	
24	
27	
30	
33	
36	
39	
42	
45	

Complete the table to the right of the graph

As you work on the problems of this investigation, look for answers to the following questions:

How can you interpret and estimate or calculate values of expressions involving fractional or decimal exponents?
How can you interpret and estimate or calculate the half-life of a substance that decays exponentially?

1. Medical scientists often are interested in the time it takes for a drug to be reduced to one half of the original dose. They call this time the **half-life** of the drug. What appears to be the half-life of insulin in this case?

2. The pattern of decay shown on this graph for insulin can be modeled well by the function $y = 10(0.95^x)$, where x is the number of minutes since the insulin entered the bloodstream.

a. Use your calculator or computer software to see how well a table of values and graph of this rule matches the pattern in the graph

b. What do the numbers 10 and 0.95 tell about the amount of insulin in the bloodstream?

c. Based on the function modeling insulin decay, what percent of active insulin is actually used up with each passing minute?

d. What recursive rule shows how the amount of insulin in the blood changes from one minute to the next, once 10 units have entered the bloodstream?

4. The insulin decay graph shows data points for three-minute intervals following the original insulin level. But the curve connecting those points reminds us that the insulin breakdown does not occur in sudden bursts at the end of each minute! It occurs *continuously* as time passes.

a. What would each of the following calculations tell about the insulin decay situation? Based on the graph, what would you expect as reasonable values for those calculations?

a. $y = 10(0.95)^{1.5}$ **b.** $y = 10(0.95)^{4.5}$ **c.** $y = 10(0.95)^{18.75}$

5. Mathematicians have figured out ways to do calculations with fractional or decimal exponents so that the results fit into the pattern for whole number exponents. One of those methods is built into your graphing calculator or computer software.

a. Enter the function $y = 10(0.95^x)$ in your calculator or computer software. Then complete a copy of the following table of values showing the insulin decay pattern at times other than whole-minute intervals.

Elapsed Time (in minutes)	0	1.5	4.5	7.5	10.5	13.5	16.5	19.5
Units of Insulin in Blood	10							

b. Compare the entries in this table with data shown by points on the graph on the preceding page.

c. Study tables and graphs of your function to estimate, to the nearest tenth of a minute, solutions for the following equations and inequality. In each case, be prepared to explain what the solution tells about decay of insulin.

i. $2 = 10(0.95^x)$ **ii.** $8 = 10(0.95^x)$ **iii.** $10(0.95^x) > 1.6$

6a. Use the function $y = 10(0.95^x)$ to estimate the half-life of insulin for an initial dose of 10 units.

b. Then estimate the half-life in cases when the initial dose is 15 units.

c. Then estimate the half-life in cases when the initial dose is 20 units.

d. Then estimate the half-life in cases when the initial dose is 25 units.

When you study a situation in which data suggest a dependent variable decreasing in value as a related independent variable increases, there are two strategies for finding a good algebraic model of the relationship. In some cases, it is possible to use the problem conditions and reasoning to determine the type of function that will match dependent to independent variable values. In other cases, some trial-and-error exploration or use of calculator or computer curve-fitting software will be necessary before an appropriate model is apparent.

From a scientific point of view, it is always preferable to have some logical explanation for choice of a model. Then the experimental work is supported by understanding of the relationship being studied. As you work on the following problems, look for answers to these questions:

What clues in problem conditions are helpful in deriving function models for experimental data involving decay?

How can logical analysis of an experiment be used as a check of a function model produced by your calculator or computer curve-fitting software?

1. Suppose that you were asked to conduct this experiment:

- Get a collection of 100 coins, shake them well, and drop them on a tabletop.
- Remove all coins that are lying heads up and record the number of coins left.
- Repeat the shake-drop-remove-record process until 5 or fewer coins remain.

Drop #	0	1	2	3	4	5	6	7	8
Coins Left	100	49	22	9	6	4	2	1	1

ai. If you were to record the results of this experiment in a table of (*drop number, coins left*) values, what pattern would you expect in the data?

aii. What function rule would probably be the best model relating drop number *n* to number of coins left *c*?

b. Conduct the experiment, record the data, and then use your calculator or curve-fitting software to find a function model that seems to fit the data pattern well.

c. Compare the model suggested by logical analysis of the experiment to that found by fitting a function to actual data. Decide which you think is the better model of the experiment and be prepared to explain your choice.

2. Suppose that the experiment in Problem 1 is modified in this way:

- Get a collection of 100 coins and place them on a table top.

- Roll a six-sided die and remove the number of coins equal to the number on the top face of the die. Record the number of coins remaining. For example, if the first roll shows 4 dots on the top of the die, remove four coins, leaving 96 coins still on the table.

- Repeat the roll-remove-record process until 10 or fewer coins remain

Drop #	0	1	2	3	4	5	6	7	8	9	10
Coins Left	100	98	93	89	86	83	79	75	69	68	67

ai. If you were to record the results of this experiment in a table of (*roll number*, *coins left*) values, what pattern would you expect in that data?

aii. What function rule would probably be the best model relating roll number n to number of coins left c?

b. Conduct the experiment, record the data, and then use your calculator or curve-fitting software to find a function model that seems to fit the data pattern well.

c. Compare the model suggested by logical analysis of the experiment to that found by fitting a function to actual data. Decide which you think is the better model of the experiment and be prepared to explain your choice.

3a. How are the data from the experiments in Problems 1 and 2 and the best-fitting function models for those data different?

3b. Why are those differences reasonable, in light of differences in the nature of the experiments that were conducted?

In studying the rebound height of a bouncing ball, you calculated powers of the fraction $\frac{2}{3}$. You can calculate a power like $\left(\frac{2}{3}\right)^4$ by repeated multiplication $\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$. But there is a shortcut rule for such calculations with exponents.

As you work on the problems in this investigation, make notes of answers to this question:

What exponent properties provide shortcut rules for calculating powers of fractions, quotients of powers, and negative exponents?

Powers of a Fraction As you work on the next calculations, look for a pattern suggesting ways to write powers of fractions in useful equivalent forms

1. Simplify

a. $\left(\frac{3}{5}\right)^3 =$

b. $\left(\frac{c}{5}\right)^2 =$

c. $\left(\frac{4}{n}\right)^5 =$

d. $\left(\frac{c^2}{5n}\right)^3 =$

2. Examine the results of your work on Problem 1.

a. What pattern seems to relate task and result in every case?

b. How would you use the definition of exponent or other reasoning to convince another student that your answer to Part a is correct?

c. What would you expect to see as the most common errors in evaluating powers of a fraction like $\left(\frac{3}{5}\right)^4 =$? Explain how you would help someone who made those errors correct their understanding of how exponents work.

Quotients of Powers Since many useful algebraic functions require division of quantities, it is helpful to be able to simplify expressions involving quotients of powers like $\frac{b^x}{b^y}$.

3. Simplify

a. $\frac{2^{10}}{2^3}=$

b. $\frac{3^6}{3^2}=$

c. $\frac{10^9}{10^3}=$

d. $\frac{2^x}{2^5}=$

e. $\frac{7^x}{7^y}=$

f. $\frac{3^5}{3^5}=$

g. $\frac{b^5}{b^3}=$

h. $\frac{b^x}{b^x}=$

4. Examine the results of your work on Problem 3.

a. What pattern seems to relate task and result in every case?

b. How would you use the definition of exponent or other reasoning to convince another student that your answer to Part a is correct?

c. What would you expect to see as the most common errors in evaluating powers of a fraction like $\frac{8^{12}}{8^4}=$? Explain how you would help someone who made those errors correct their understanding of how exponents work.

Negative Exponents Suppose that you were hired as a science lab assistant to monitor an ongoing experiment studying the growth of an insect population. If the population when you took over was 48 and it was expected to double every day, you could estimate the population for any time in the future or the past with the function $p = 48(2^x)$.

Future estimates are easy: One day from now, the population should be about $48(2^1) = 96$; two days from now it should be about $48(2^2) = 48(2)(2) = 192$, and so on.

Estimates of the insect numbers in the population before you took over require division: One day earlier, the population should have been about

$$\begin{aligned} 48\left(2^{-2}\right) &= (48 \div 2) \div 2 \\ &= 48 \div 2^2 \\ &= 48\left(\frac{1}{2}\right)^2 \\ &= 12 \end{aligned}$$

This kind of reasoning about exponential growth suggests a general rule that for any nonzero number b and any integer, $b^{-n} = \frac{1}{b^n}$

6. The rule for operating with negative integer exponents also follows logically from the property about quotients of powers and the definition $b^0 = 1$. Justify each step in the reasoning below.

$$\begin{aligned} \frac{1}{b^n} &= \frac{b^0}{b^n} \\ &= b^{0-n} \\ &= b^{-n} \end{aligned}$$

7. Use the relationship between fractions and negative integer exponents to write each of the following expressions in a different but equivalent form. In Parts a-f, write an equivalent fraction that does not use exponents at all.

a. $5^{-3} =$ b. $6^{-1} =$ c. $2^{-4} =$

d. $x^{-3} =$ e. $\left(\frac{1}{2}\right)^{-3} =$ f. $\left(\frac{2}{5}\right)^{-2} =$

g. $\left(\frac{2}{5}\right)^{-1} =$ h. $\frac{1}{a^4} =$

8. Examine the results of your work on Problem 7.

a. What pattern seems to relate task and result in every case?

b. How would you use the definition of exponent or other reasoning to convince another student that your answer to Part a is correct?

c. What would you expect to see as the most common errors in evaluating powers of a fraction like $\left(\frac{4}{3}\right)^{-2} =$? Explain how you would help someone who made those errors correct their understanding of how exponents work.

In your work on problems of insulin decay, you found that some questions required calculation with exponential expressions involving a fractional base and fractional powers. For example, estimating the amount of insulin active in the bloodstream 1.5 minutes after a 10-unit injection required calculating $10(0.95^{1.5})$.

Among the most useful expressions with fractional exponents are those with power one-half. It turns out that one-half powers are connected to the square roots that are so useful in geometric calculations like those involving the Pythagorean Theorem.

For any non-negative number b, $b^{\frac{1}{2}} = \sqrt{b}$

Expressions like $\sqrt{b}$, $\sqrt{5}$, and $\sqrt{9 - x^2}$ are called radicals. As you work on the following problems, keep this question in mind:

How can you use your understanding of properties of exponents to guide your thinking about one-half powers, square roots, radical expressions, and rules for operating with them?

1. For integer exponents m and n, you know that $(a^m)^n = a^{mn}$. That property can be extended to work with fractional exponents.

a. Write each of these expressions in standard number form without exponents or radicals.

i. $\left(2^{\frac{1}{2}}\right)^2 =$

ii. $\left(5^{\frac{1}{2}}\right)^2 =$

iii. $\left(12^{\frac{1}{2}}\right)^2 =$

iv. $\left(2.4^{\frac{1}{2}}\right)^2 =$

2. Write each of the following expressions in an equivalent form using radicals and then in simplest number form (without exponents or radicals).

i. $(25)^{\frac{1}{2}} =$

ii. $(9)^{\frac{1}{2}} =$

iii. $\left(\frac{9}{4}\right)^{\frac{1}{2}} =$

iv. $(100)^{\frac{1}{2}} =$

3. Use the properties of square roots in Problem 4 to write expressions a-h in several equivalent forms. In each case, try to find the simplest equivalent form—one that involves only one radical and the smallest possible number inside that radical. Check your ideas with calculator estimates of each form. For example,

$$\begin{aligned}\sqrt{48} &= \sqrt{16}\sqrt{3} \\ &= 4\sqrt{3}\end{aligned}$$

Calculator estimates show that $\sqrt{48} \approx 6.93$ and $4\sqrt{3} \approx 6.93$.

a. $\sqrt{45}$

b. $\sqrt{27}$

c. $\sqrt{32}$

d. $\sqrt{12}$

b. $\sqrt{18}$

c. $\sqrt{96}$